# Relativity and Symmetry

Relativity : The laws of should take the same form to all observers in inertial frames.

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Isotropy of space : If we take our "lab" and rotate it, nothing should change.

Homogeneity of space : If we take our "lab" and translate it, nothing should change.

Homogeneity of time : The laws of physics yesterday are the same as today and tomorrow.

* "lab" literally means any physical thing that influences an experiment!

Galileo / Newton → Absolute Time

Einstein → Constancy of speed of light

Absolute Time ⟵ incompatible ⟶ Constancy of speed of light

$\Downarrow$

Homogeneity and Isotropy of Spacetime

What is meant by the laws of physics are the same? In the Newtonian case we can consider $\sum_j \vec{F}_{jb} = m_b \vec{a}_b$. This will not suffice to describe observations made by an accelerating observer.

If we take the action $S$ as a fundamental definition of our theory, then it should be invariant.

Invariance of $S \Rightarrow$ covariance of e.o.m.

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Symmetry is an incredibly powerful tool to help simplify calculations.
But, symmetry also plays a more fundamental role in determining the type of dynamics in certain physical theories.
For example special relativity is nothing more than a statement about the symmetries of physics on a particular spacetime.
Additionally, the fundamental interactions can be understood as arising from symmetry principles (gauge invariance).

Your first exposure to symmetry was probably <u>static</u> type, e.g. $\triangle \xRightarrow{120^\circ} \triangle$ (geometry, shapes, etc.)

Static symmetries are easy to visualize, but ...
We will be more interested in <u>dynamical</u> symmetries, e.g. Lagrangian: $L \xRightarrow{\text{some transformation}} L' = L$

No matter what type of symmetry we consider, the spirit is the same, i.e. we enact a <u>transformation</u> on something and afterwards that something looks the same.

Now naively it may seem that our something must only be built out of things which themselves are invariant. If this were the case it would be terribly restrictive. Fortunately, we can build an invariant something out of pieces which are not invariant so long as we combine them in an appropriate way, e.g. we can build a rotationally invariant scalar from vector components with a dot product!

So our preliminary focus will be on describing transformations. We will come back to making sure they are symmetries of a Lagrangian a bit later.

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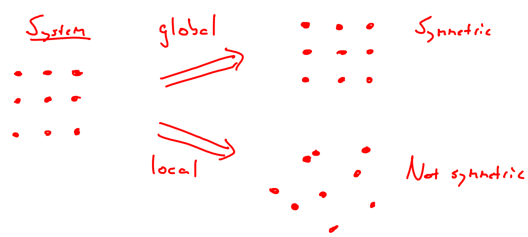
Transformations come in many different types: global, local, discrete, continuous, finite, infinite, compact, non-compact, internal, spacetime

To clarify most of these words we will look at static symmetry examples:

## Global vs. Local

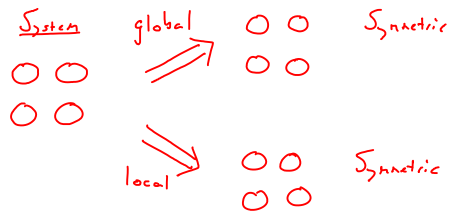
Example 1:

Transformation = translate each dot




Example 2:

Transformation = rotate each circle in plane




Note: If a system is symmetric under local transformations then it is automatically symmetric under global transformations, but the reverse is not true!

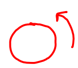
## Discrete vs. Continuous

Discrete (finite or infinite)

Example 1:  $R_{120°}$ finite since $\{1, R_{120}, R_{240}\}$

Example 2: $-\infty \leftarrow \cdots \rightarrow +\infty$ ($T_1$, $T_2$) infinite since $\{T_1, T_2, T_3, \cdots\}$

Continuous (compact or noncompact)

Example 1:  $R_\theta$ $\theta \in [0, 2\pi)$ compact

Example 2: $-\infty \leftarrow \xrightarrow{T_d} +\infty$ $d \in (-\infty, \infty)$ noncompact

## Spacetime vs. Internal

If we coordinatize spacetime, then spacetime transformations also change coordinates while internal transformations do nothing to the coordinates.

Note: Special Relativity is associated with spacetime symmetries.
The strong, weak and electromagnetic forces are associated with internal symmetries.

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For our purposes we can treat transformations mathematically using the concepts of groups and representations.

A group $G$ is a collection of transformations $\{A, B, \cdots\}$ with a composition $\bullet$ that satisfies:

1. Closure - if $A, B \in G \Rightarrow A \bullet B \in G$
2. Identity - there is some $I \in G$ such that $I \bullet A = A$ for any $A \in G$
3. Inverse - for any $A \in G$ there is an $A^{-1} \in G$ such that $A^{-1} \bullet A = I$
4. Associativity - $A \bullet (B \bullet C) = (A \bullet B) \bullet C$

These will be very important in building invariants!

We could add commutativity, i.e. $A \bullet B = B \bullet A$, in which case we have an abelian group, but we actually need groups that don't commute, i.e. they are non-abelian.

If we can take a subset of the elements of a group and they form a group themselves then this is a subgroup of the original group. Note: subgroups always have to include the identity and inverses and have to be careful to remain closed!

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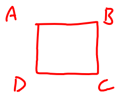
We can abstractly specify a group, e.g. Rotations in 2D with composition that we add the rotation angles.

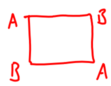
But more often we think (and calculate) in terms of how the group transformations act on things. These are called <u>representations</u> of the group.

A single group can often have many different representations. Some are more useful since they fully illustrate the content of the group, these are called <u>faithful representations</u>.

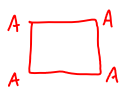
Example: $G$ = Rotations in plane by $90^\circ$ w/ usual composition (addition of angles).

Rep $r_1$: 

$\{ I, R_{90}, R_{180}, R_{270} \}$ This is the only faithful representation!

Rep $r_2$: 

$\{ I, R_{90} \}$ This is a degenerate representation.

Rep $r_3$: 

$\{ I \}$ This is the highly degenerate identity representation.

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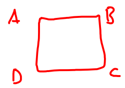
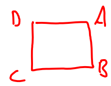
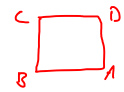
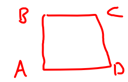
We can often work with representations where the transformations act linearly using matrices:

If:  $= \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$  $= \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$  $= \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$  $= \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

then $I = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$, $R_{90} = \begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$, $R_{180} = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$, $R_{240} = \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}$

Notice that the matrices behave as expected, e.g. $R \circ R_{180} = R_{240}$, etc.

Note: This is an abelian group since any $R R' = R' R$.